The lattice of integer partitions and its infinite extension

Matthieu Latapy and Ha Duong Phan [1]

March 1999

Abstract : In this paper, we use a simple discrete dynamical system to study integers partitions and their lattice. The set of reachable configurations equiped with the order induced by the transitions of the system is exactly the lattice of integer partitions equiped with the dominance ordering. We first explain how this lattice can be constructed, by showing its self-similarity property. Then, we define a natural extension of the system to infinity. Using a self-similar tree, we obtain an efficient coding of the obtained lattice. This approach gives an interesting recursive formula for the number of partitions of an integer, where no closed formula have ever been found. It also gives informations on special sets of partitions, such as length bounded partitions.

1 Preliminaries

A partition of the integer n is a k-tuple $s = (s_1, s_2, \dots, s_k)$ of positive integers such that $\sum_{i=1}^{k} s_i = n$ and $s_i \geq s_{i+1}$ for all i between 1 and k (with the assumption that $s_{k+1} = 0$). The Ferrers diagram of a partition $s = (s_1, s_2, \dots, s_k)$ is a drawing of s on k adjacent columns such that the i-th column is a pile of s_i packed squares (called grains). See Figure 1 for examples.

The *dominance ordering* [DP90] is defined in the following way. Consider two partitions of the integer n: $s = (s_1, s_2, \dots, s_k)$ and $t = (t_1, t_2, \dots, t_l)$. Then

$$s \geq t \text{ if and only if } \sum_{i=1}^{j} s_i \geq \sum_{i=1}^{j} t_i \text{ for all } j,$$

i.e. the prefix sums of s are greater than or equal to the prefix sums of t.

From [Bry73], it is known that the set of partitions of an integer n equiped with the dominance ordering is a lattice denoted by $L_B(n)$. In this paper, Brylawski proposed a dynamical approach to study this lattice. We will introduce a few notations to explain it intuitively. For more details about integer partitions, we refer to [And76].

Let $s = (s_1, \dots s_k)$ be a partition. The *height difference of s at i*, denoted by $d_i(s)$, is the integer $s_i - s_{i+1}$ (with the assumption that $s_{k+1} = 0$). We say that the partition s has a *cliff* at i if $d_i(s) \geq 2$. We say that s has a *slippery plateau* at i if there exists $k > i$ such that $d_j(s) = 0$ for all $i \leq j < k$ and $d_k(s) = 1$. The integer $k - i$ is then called the *length* of the slippery plateau at i. Likewise, s has a *non-slippery plateau* at i if $d_j(s) = 0$ for all $i \leq j < k$ and it has a cliff at k. The integer $k - i$ is called the *length* of the non-slippery plateau at i. The partition s has a *slippery step* at i if the partition defined by $s' = (s_1, \dots, s_i - 1, \dots, s_k)$ (if it exists) has a slippery plateau at i. Likewise, s has a *non-slippery step* at i if s' has a non-slippery plateau at i. See Figure 1.

Consider now the partition $s = (s_1, s_2, \dots, s_k)$. Brylawski defined the two following evolution rules: one grain can fall from column i to column $i + 1$ if s has a cliff at i,

[1] LIAFA, Université Paris 7, 2 place Jussieu, 75005 Paris. (latapy,phan)@liafa.jussieu.fr

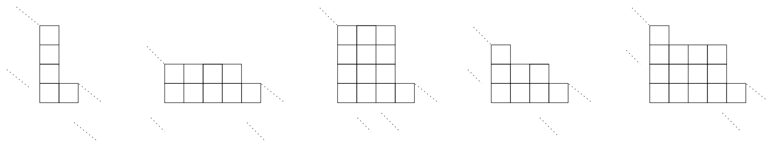

Figure 1: From left to right: a cliff, a slippery plateau (length 3), a non-slippery plateau (length 2), a slippery step (length 2) and a non-slippery step (length 3).

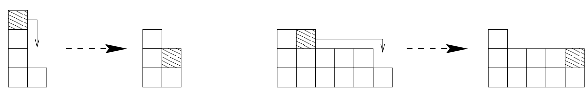

Figure 2: The two evolution rules of the dynamical system

and one grain can slip from column i to column $i + l + 1$ if s has a slippery step of length l at i. See Figure 2.

Such a fall or a slip is called a *transition* of the system and is denoted by $s \xrightarrow{i} t$ where i is the column from which the grain falls or slips. If one starts from the partition (n) and iterates this dynamical process, one obtains all the partitions of n, and the dominance ordering is nothing but the reflexive and transitive closure of the relation induced by the transition rule [Bry73]. See Figure 3 for examples with $n = 7$ and $n = 8$. Then, the set of directly reachable configurations from s, denoted by $dirreach(s)$, is the set $\{t \mid s \xrightarrow{i} t\}$. Notice that in the context of dynamical systems theory, those elements are called *the immediate successors* of s. However, since we are concerned here with orders theory, we can not use this term, which takes another meaning in this context.

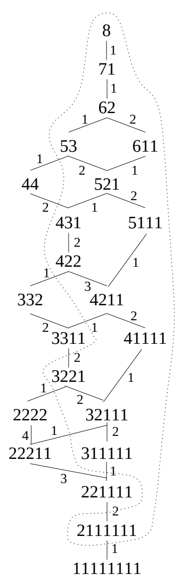

Figure 3: Diagrams of the lattices $L_B(n)$ for $n = 7$ and $n = 8$. As we will see, the set $L_B(7)$ is isomorphic to a sublattice of $L_B(8)$. On the diagram of $L_B(8)$, we included in a dotted line this sublattice.

Before entering in the core of the topic, we need one more notation. If the k-tuple $s = (s_1, s_2, \ldots, s_k)$ is a partition, then the k-tuple $(s_1, s_2, \ldots, s_{i-1}, s_i + 1, s_{i+1}, \ldots, s_k)$

is denoted by $s^{\downarrow i}$. In other words, $s^{\downarrow i}$ is obtained from s by adding one grain on its i-th column. Notice that the k-tuple obtained this way is not necessarily a partition. Finally, if S is a set of partitions, then $S^{\downarrow i}$ denotes the set $\{s^{\downarrow i} | s \in S\}$.

We will now study the structure of the lattice of the partitions of an integer n and we will show its self-similarity. Then, we will define a natural infinite extension of these lattices: the lattice $L_B(\infty)$ of all the partitions of any integer. This lattice has interesting properties, which we will analyse. Finally, we will construct a tree based on the self-similarity observed over the previously studied lattices. This tree represents the duplication process detailed in the beginning of the paper. It has a recursive structure, from which we will obtain a recursive formula for the number of partitions of one integer n and some results about certain classes of partitions. Notice that this kind of ideas has first been developped in the special case of SPM (that is a restriction to its only first rule of the dynamical system presented here) [LMMP98].

2 From $L_B(n)$ to $L_B(n+1)$

In this section, our aim is to construct (the graph of) the transitive reduction of $L_B(n+1)$ from the one of $L_B(n)$ with the edges of this graph labelled by the column from which the grain falls or slips (See Figure 3). We will call *construction of a lattice* the computation of the labelled graph of its transitive reduction. We first show that $L_B(n)^{\downarrow 1}$ is a sublattice of $L_B(n+1)$. For example, in Figure 3 we included in a dotted line $L_B(7)^{\downarrow 1}$ within $L_B(8)$. The use of this remark allows us to construct $L_B(n+1)$ from $L_B(n)$ by adding to $L_B(n)^{\downarrow 1}$ the missing elements of $L_B(n+1)$. After characterizing those elements that must be added, we obtain a simple and efficient algorithm (given in Appendix A) to acheive the construction of $L_B(n+1)$ from $L_B(n)$.

Proposition 1 *$L_B(n)^{\downarrow 1}$ is a sublattice of $L_B(n+1)$.*

Proof : We must show that:
- if $\inf(a,b) = c$ in $L_B(n)$ then $\inf(a^{\downarrow 1}, b^{\downarrow 1}) = c^{\downarrow 1}$ in $L_B(n+1)$,
- if $\sup(a,b) = c$ in $L_B(n)$ then $\sup(a^{\downarrow 1}, b^{\downarrow 1}) = c^{\downarrow 1}$ in $L_B(n+1)$.

Recall that the dominance ordering over $L_B(n)$ implies [Bry73]:

$$\inf(a,b) = c \text{ iff for all } j \text{ one has } \sum_{i=1}^{j} c_i = min(\sum_{i=1}^{j} a_i, \sum_{i=1}^{j} b_i).$$

From this, it is obvious that $c^{\downarrow 1} = \inf(a^{\downarrow 1}, b^{\downarrow 1})$ and it is straightforward to see that $c^{\downarrow 1}$ is in $L_B(n+1)$.

Let now c be the $\sup(a,b)$ in $L_B(n)$ and d be the $\sup(a^{\downarrow 1}, b^{\downarrow 1})$ in $L_B(n)^{\downarrow 1}$. We will show that $d = c^{\downarrow 1}$. We have $c \geq a$ and $c \geq b$, therefore $c^{\downarrow 1} \geq a^{\downarrow 1}$ and $c^{\downarrow 1} \geq b^{\downarrow 1}$. This implies that $c^{\downarrow 1} \geq d$. For showing that $d \geq c^{\downarrow 1}$, let us begin by showing that $d_1 = c_1 + 1$. We can suppose $a_1 \geq b_1$. The partition $(a_1, a_1, a_1 - 1, a_1 - 2, \dots)$ is greater than a and b, hence it is greater than c. Moreover, $c \geq a$ implies $c_1 \geq a_1$ hence $c_1 = a_1$. Since $a^{\downarrow 1} \leq d \leq c^{\downarrow 1}$, we then have $d_1 = a_1 + 1 = c_1 + 1$. Let $e = (d_1 - 1, d_2, d_3, \dots)$. Since $d \leq c^{\downarrow 1}$ and $c_1 = a_1$, e is a valid partition ($d_1 - 1 \geq d_2$). Moreover, $d \geq a^{\downarrow 1}$ and $d \geq b^{\downarrow 1}$, hence $e \geq a$ and $e \geq b$. This implies that $e \geq \sup(a,b) = c$ and that $d \geq c^{\downarrow 1}$, which ends the proof. □

This result shows that one can construct the lattice $L_B(n+1)$ from $L_B(n)$. The first step of this construction is to construct the set $L_B(n)^{\downarrow 1}$ by adding one grain to the first column of every element of $L_B(n)$. Then, we have to add the missing elements

and their covering relations. Therefore, we will now consider the consequences of the addition of one grain on the first column of a partition, depending on its structure.

Proposition 2 *Let s be a partition. Then, we have the following statements:*

1. *if s has a cliff or a non-slippery plateau at 1 then:*
$$s \xrightarrow{i} t \Longleftrightarrow s^{\downarrow_1} \xrightarrow{i} t^{\downarrow_1}$$
2. *if s has a slippery plateau of length l at 1 then $s^{\downarrow_1} \xrightarrow{1} s^{\downarrow_{l+1}}$ and:*
$$dirreach(s^{\downarrow_1}) = dirreach(s)^{\downarrow_1} \cup \{s^{\downarrow_{l+1}}\}$$
3. *if s has a slippery step at 1, then let t be such that $s \xrightarrow{1} t$; we have then $s^{\downarrow_1} \xrightarrow{1} s^{\downarrow_2} \xrightarrow{2} t^{\downarrow_1}$ and:*
$$dirreach(s^{\downarrow_1}) = (dirreach(s)^{\downarrow_1} \setminus \{t\}) \cup \{s^{\downarrow_2}\}$$

Proof :

1. Notice first that the transitions possibles from s on columns other than the first are still possible from $s^{\downarrow_1}$. Another remark is that the added grain can not prevent any transition. It suffices now to see that the addition of one grain on a cliff does not allow a new transition from the first column, since such a transition was already possible. Likewise, the addition of one grain on a non-slippery plateau does not allow a new transition.

2. No transition from s was possible on its first column. However, $s^{\downarrow_1}$ has a slippery step at 1 and so a transition $s^{\downarrow_1} \xrightarrow{1} t$ is now possible. It is easy to verify that $t = s^{\downarrow_{l+1}}$.

3. Notice first that t is a partition, since s has a slippery step at 1. Moreover, transitions on columns other than the first are not affected by addition of one grain on the first column. Let us now observe what happens about the transition from the first column. Notice that one transition from the first column is still possible, since $s^{\downarrow_1}$ has a cliff at column 1. However, the transition $s^{\downarrow_1} \xrightarrow{1} t^{\downarrow_1}$ is now impossible, but a new transition is possible: $s^{\downarrow_1} \xrightarrow{1} s^{\downarrow_2}$. Now, $s^{\downarrow_2}$ has either a slippery step or a cliff at 2, hence $t^{\downarrow_1}$ is directly reachable from it. □

For a given integer n, let us denote by S the set of partitions of n with a slippery step at 1, by T the set of partitions of n with a non-slippery step at 1, and by U_l the set of partitions of n with a slippery plateau of length l at 1. The propositions above show that, once we have $L_B(n)^{\downarrow_1}$, the next step of the construction is the addition of the elements and edges of the sets $S^{\downarrow_2}$, $T^{\downarrow_2}$ and $U_l^{\downarrow_{l+1}}$. Now, we must add the missing transitions from these new elements, and the missing elements directly reachable from them. We show below that indeed no element is missing, and we give a description of which missing transitions need to be added.

Theorem 1 *Every element of $L_B(n+1)$ is in $L_B(n)^{\downarrow_1}$, in $S^{\downarrow_2}$, in $T^{\downarrow_2}$ or in $U_l^{\downarrow_{l+1}}$ for some l.*

Proof : We have shown that all the configurations directly reachable from the elements of $L_B(n)^{\downarrow_1}$ are in this union. Let us now show that all the configurations directly reachable from the elements of the union are already in this set. Several cases are possible.

- Let $s \in S$. Then, $s \xrightarrow{1} t$ in $L_B(n)$. Only one of the possible transitions from s is affected by the addition of one grain on the second column: the transition on column 1. Moreover, due to the choice of s in S, a transition $\xrightarrow{2}$ is possible from $s^{\downarrow_2}$. From these remarks, we obtain:

$$dirreach(s^{\downarrow_2}) = (dirreach(s) \setminus \{t\})^{\downarrow_2} \;\cup\; \{t^{\downarrow_1}\}.$$

Notice now that the elements of $dirreach(s) \setminus \{t\}$ have a slippery step at 1, hence the elements from $(dirreach(s) \setminus \{t\})^{\downarrow_2}$ have already been added. Moreover, $t^{\downarrow_1}$ is in $L_B(n)^{\downarrow_1}$, hence there is no missing element directly reachable from $s^{\downarrow_2}$, and no missing transition from $s^{\downarrow_2}$.

- Let $s \in T$. Then, the addition of one grain on the second column of s does not prevent any transition, and we have:

$$dirreach(s^{\downarrow_2}) = dirreach(s)^{\downarrow_2}.$$

Moreover, the elements of $dirreach(s)$ have a slippery or non-slippery step at 1, and therefore the elements of $dirreach(s)^{\downarrow_2}$ and the transitions from $s^{\downarrow_2}$ have already been added.

- Let $s \in U_l$. This case requires more attention. We distinguish three subcases:

  1. s has a cliff at $l+1$. Then, we have $s \xrightarrow{l+1} t \xrightarrow{l} u$ in $L_B(n)$. The addition of one grain on the $(l+1)$-th column of s does not prevent any transition, hence we have:
  $$dirreach(s^{\downarrow_{l+1}}) = dirreach(s)^{\downarrow_{l+1}}.$$
  From the choice of s, we know that the elements of $dirreach(s) \setminus \{t\}$ have a slippery plateau at 1. Therefore all the elements of $dirreach(s)^{\downarrow_{l+1}}$ have already been added. Moreover, one can verify that, if $s = (s_1, \dots)$, then $u = (s_1, \dots, s_l - 1, s_{l+1}, s_{l+2}+1, \dots)$, and that $s^{\downarrow_1} \xrightarrow{1} s^{\downarrow_{l+1}} \xrightarrow{l+1} u^{\downarrow_l}$. Therefore, $t^{\downarrow_{l+1}} = u^{\downarrow_l}$. Moreover, u has a slippery plateau of length $l-1$ at 1, hence the element $u^{\downarrow_l}$ has already been added. Thus, no element is missing; there is only one missing transition: $s^{\downarrow_{l+1}} \xrightarrow{l+1} t^{\downarrow_{l+1}}$.

  2. s has a non-slippery step at l, hence a non-slippery plateau of length l' at $l+1$ (with possibly $l' = 0$). Then, the addition of one grain on the $(l+1)$-th column of s does not prevent any transition that was previously possible. Therefore, we have:
  $$dirreach(s^{\downarrow_{l+1}}) = dirreach(s)^{\downarrow_{l+1}}.$$
  The elements of $dirreach(s)$ all have a non-slippery plateau at 1, hence all the elements of $dirreach(s)^{\downarrow_{l+1}}$ have already been added.

  3. s has a slippery step of length l' at l, hence a slippery plateau of length l' at $l+1$ ($l' = 0$ is possible). Then, $s \xrightarrow{l} t$ in $L_B(n)$. The possible transitions from $s^{\downarrow_{l+1}}$ are the same as the possible ones from s, except the transition on the column l. All the elements directly reachable from s except t have a slippery plateau at 1, hence the elements of $dirreach(s)\setminus\{t\}^{\downarrow_{l+1}}$ have already been added. Moreover, $s^{\downarrow_{l+1}} \xrightarrow{l+1} s^{\downarrow_{l+l'+1}}$. But we can verify that $s^{\downarrow_{l+l'+1}} = t^{\downarrow_l}$, and, since t has a slippery plateau of length $l-1$ at 1, this element has already been added; there is only one missing transition: $s^{\downarrow_{l+1}} \xrightarrow{l+1} t^{\downarrow_l}$. $\square$

It is now straightforward to realize that the algorithm given in Appendix A constructs the lattice $L_B(n+1)$ from $L_B(n)$. Notice that we can obtain $L_B(n)$ for an arbitrary integer n by starting from $L_B(0)$ and iterating this algorithm. We show in Appendix A that the complexity of this algorithm is linear with respect to the number of added elements and transitions, and hence we have an algorithm that constructs $L_B(n)$ in linear time with respect to the size of $L_B(n)$.

3 The infinite lattice $L_B(\infty)$

We will now define $L_B(\infty)$ as the set of all configurations reachable from (∞) (this is the configuration where the first column contains infinitely many grains and all the other columns contain no grain). As in the previous section, the dominance ordering on $L_B(\infty)$ (when the first component, always equal to ∞, is ignored) is equivalent to the order induced by the dynamical system. Notice that any element s of $L_B(\infty)$ has the form $(\infty, s_2, s_3, \ldots, s_k)$. The first partitions in $L_B(\infty)$ are given in Figure 4 along with their convering relations (the first column, which always contains an infinite number of grains, is not represented on this diagram).

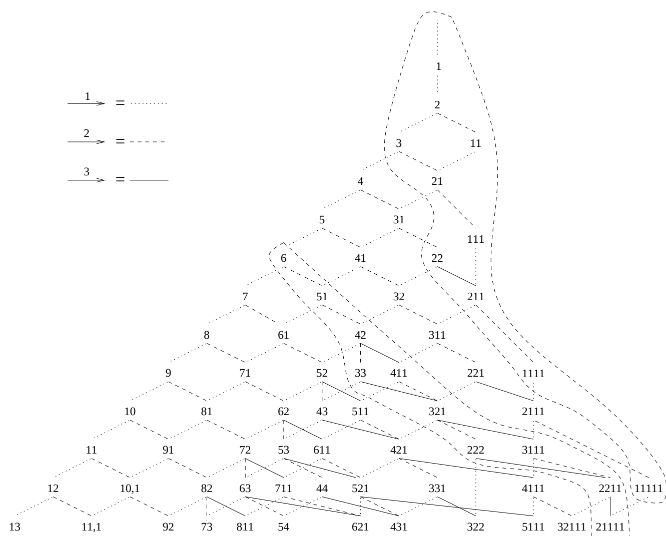

Figure 4: First elements and transitions of $L_B(\infty)$. As shown on this figure for $n = 6$, we will see two ways to find parts of $L_B(\infty)$ isomorphic to $L_B(n)$ for any n.

It is easy to see that we have a characterisation of the order similar to the one given in [Bry73] for the finite case: let s and t be two elements of $L_B(\infty)$. Then, $s \geq_\infty t$ iff for all j such that $2 \leq j \leq \max(p, q)$, $\sum_{i \geq j} s_i \leq \sum_{i \geq j} t_i$.

Theorem 2 *The set $L_B(\infty)$ is a lattice. Moreover, if we consider $s = (\infty, s_2, \ldots, s_k)$ and $t = (\infty, t_2, \ldots, t_l)$ two elements of $L_B(\infty)$, then, $\inf_\infty(s,t) = u$ in $L_B(\infty)$, where*

$$u_i = max(\sum_{j \geq i} s_j, \sum_{j \geq i} t_j) - \sum_{j > i} u_j \quad \textit{for all } i \textit{ such that } 2 \leq i \leq max(k, l)$$

and $\sup_\infty(s,t) = \inf\{u \in L_B(\infty), u \geq_\infty s, u \geq_\infty t\}$.

Proof : Let s and t be two elements of $L_B(\infty)$ and let u be the composition defined as above, we first prove that u is an element of $L_B(\infty)$ and then we prove that u is

equal to $inf_\infty(s,t)$. Let $n = 2(\sum_{i\geq 2} s_i + \sum_{i\geq 2} t_i)$. Let $s' = (n - \sum_{i\geq 2} s_i, s_2, \dots, s_k)$, $t' = (n - \sum_{i\geq 2} t_i, t_2, \dots, t_l)$ and $u' = (n - \sum_{i\geq 2} u_i, u_2, \dots, u_{max(k,l)})$. It is then obvious that s' and t' are two partitions of n and that u' is the meet of s' and t' by dominance ordering in $L_B(n)$. Therefore, u' is a decreasing sequence, hence u is an element of $L_B(\infty)$. Moreover, according to the definition of $\geq_\infty$, u is the maximal element of $L_B(\infty)$ which is smaller than s and t, so $u = inf_\infty(s,t)$.

By definition, $L_B(\infty)$ has a maximal element and is closed for the meet; then $L_B(\infty)$ is a lattice. □

Theorem 3 *Let n be a positive integer. The application:*

$$\begin{array}{cccc} \pi: & L_B(n) & \longrightarrow & L_B(\infty) \\ & s = (s_1, s_2, \cdots, s_k) & \longrightarrow & \bar{s} = (\infty, s_2, \dots, s_k) \end{array}$$

is a lattice embedding.

Proof : It is obvious that π is injective. Moreover, we can apply a similar proof to the one of Proposition 1 to show that $inf_\infty(\pi(s), \pi(t)) = \pi(inf_{L_B(n)}(s,t))$ and $sup_\infty(\pi(s), \pi(t)) = \pi(sup_{L_B(n)}(s,t))$. We can then conclude that π is a lattice embedding. □

Let $\overline{L_B(n)}$ denote the image by π of $L_B(n)$ in $L_B(\infty)$. From Theorem 3, $\overline{L_B(n)}$ is a sublattice of $L_B(\infty)$. From Proposition 1, $L_B(n)^{\downarrow_1}$ is a sublattice of $L_B(n+1)$, hence, since $\overline{L_B(n)^{\downarrow_1}} = \overline{L_B(n)}$, there is an increasing sequence of sublattices:

$$\overline{L_B(0)} \leq \overline{L_B(1)} \leq \dots \leq \overline{L_B(n)} \leq \overline{L_B(n+1)} \leq \dots \leq L_B(\infty).$$

where $\leq$ denotes the sublattice relation.

Theorem 3 gives a way to find sublattices of $L_B(\infty)$ that are isomorphic to $L_B(n)$ for any n. We will see in the following another way to find such parts. In order to acheive this goal, we first study the union of the lattices $L_B(n)$ for any n.

Let $s = (\infty, s_2, s_3, \dots, s_k)$ be an element of $L_B(\infty)$. If one takes $s_1 = s_2 + 1$ and $n = \sum_{i=1}^k s_i$, then the partition $s' = (s_1, s_2, \dots, s_k)$ is an element of $L_B(n)$. Since $s = \pi(s')$, this implies that s is an element of $\overline{L_B(n)}$, therefore:

$$\bigcup_{n\geq 0} \overline{L_B(n)} = L_B(\infty)$$

Let us now study the disjoint union (denoted by $\sqcup$) of the lattices $L_B(n)$ when n varies. As we have seen in Section 2, $L_B(n)^{\downarrow_1}$ is a sublattice of $L_B(n+1)$. Thus we can define a natural order over the set $L_B(n) \sqcup L_B(n+1)$ and extend it to the union of all the $L_B(n)$'s. Let us define

$$L = \bigsqcup_{n\geq 0} L_B(n)$$

The covering relation on L is defined as follows: let $s \in L_B(m)$ and $t \in L_B(n)$. We have $s \xrightarrow{i} t$ in L if and only if either one of the following applies:

- $n = m$ and $s \xrightarrow{i} t$ in $L_B(n)$
- $i = 0$, $n = m + 1$ and $t = s^{\downarrow_1}$.

In other terms, the elements of $L_B(n)$ are linked to each other as usual, whereas each element s of $L_B(n)$ is linked to $s^{\downarrow_1} \in L_B(n+1)$ by an edge labelled with 0.

From this, one can introduce an order on the set L in the usual sense, by defining it as the reflexive and transitive closure of this covering relation. We now show that $L_B(\infty)$ is isomorphic to L, and hence that L is a lattice.

Theorem 4 *The application χ defined by:*

$$\chi : \bigsqcup_{n \geq 0} L_B(n) \longrightarrow L_B(\infty)$$

$$s = (s_1, s_2, \dots, s_k) \mapsto \chi(s) = (\infty, s_1, s_2, \dots, s_k)$$

is a lattice isomorphism. Moreover, $s \xrightarrow{i} t$ in $\bigsqcup_{n \geq 0} L_B(n)$ if and only if $\chi(s) \xrightarrow{i+1} \chi(t)$ in $L_B(\infty)$.

Proof : χ is clearly bijective. Moreover, notice that for all s, t in L, $s \xrightarrow{i} t$ if and only if $\chi(s) \xrightarrow{i+1} \chi(t)$. Therefore, χ is an order isomorphism. Since $L_B(n)$ is a lattice, this implies that χ is a lattice isomorphism. □

Due to this result, in the remainder of this paper, we represent the elements of $L_B(\infty)$ without their first column (but we keep the original labels of the edges).

Theorem 5 *For all integer n, $L_B(n)$ is a sublattice of L.*

Proof : Let s and t be two elements of $L_B(n)$, we prove that $inf_L(s,t)$ and $sup_L(s,t)$ belong to $L_B(n)$. Let u be $inf_{L_B(n)}(s,t)$ and u' be $inf_L(s,t)$. We have, $s \geq_L u' \geq_L u$, then $\sum_{i \geq 1} s_i \leq \sum_{i \geq 1} u'_i \leq \sum_{i \geq 1} u_i$, and $\sum_{i \geq 1} u'_i = n$. this implies that u' belongs to $L_B(n)$, and we obtain $u' = u$. The proof is analogous for the *sup*. □

As announced, since L is isomorphic to $L_B(\infty)$, this theorem gives us a second way to find sublattices of $L_B(\infty)$ that are isomorphic to $L_B(n)$ for any n. Our aim is now to construct parts of $L_B(\infty)$. Of course, one way to acheive this goal is to construct $L_B(n)$. However, this does not give filters [2] of $L_B(\infty)$, which is however possible. We explain how in the following.

Notice first that $L_B(\infty)$ can be viewed as a limit of the sequence of posets defined for any n by:

$$\bigsqcup_{0 \leq i \leq n} L_B(i)$$

and denoted by $L_B(\leq n)$. From Theorem 3 and 5, we can deduce an efficient method to construct $L_B(\leq n)$ for all n: it suffices to compute (recursively) $L_B(\leq n-1)$, extract from it the part which is isomorphic to $L_B(n)$, and then add the links of the set $\{s \xrightarrow{0} s^{\downarrow_1} \ s.t.\ s \in L_B(n-1)\}$. We obtain this way $L_B(\leq n)$. We show now that $L_B(\leq n)$ is a sublattice of $L_B(\infty)$ for all n, which implies directly that it is a filter of $L_B(\infty)$.

Proposition 3 *The poset $L_B(\leq n)$ is a sublattice of $L_B(\infty)$ for all n.*

Proof : To prove that $L_B(\leq n)$ is a sublattice of $L_B(\infty)$, we consider two elements s and t of $L_B(\leq n)$ and show that $inf_\infty(s,t)$ and $sup_\infty(s,t)$ belong to $L_B(\leq n)$. There exist k and l such that $s \in L_B(k)$ and $t \in L_B(l)$. We can suppose whithout loss

[2] A filter F of a poset P is a subset of P such that $\forall x \in F,\ \forall y \in P,\ y \geq x \Rightarrow y \in F$.

of generality that $k \leq l \leq n$. Let $u = sup_\infty(s,t)$. Since $u \geq_\infty s$, there exists an integer $m \leq k$ such that $u \in L_B(m)$, hence $u \in L_B(\leq n)$. Let now $v = inf_\infty(s,t)$. If $s = (s_1, s_2, \dots)$, let $s' = (s_1{+}l{-}k, s_2, \dots)$. Then, s' is in $L_B(l)$ and $inf_\infty(s',t) \in L_B(l)$. As $v = inf_\infty(s,t) \geq inf_\infty(s',t)$, we have $v \in L_B(\leq l)$. This implies the result. $\square$

4 The infinite binary tree $T_B(\infty)$

As shown in our procedure to construct $L_B(n+1)$ from $L_B(n)$, each element s of $L_B(n+1)$ is obtained from an element $s' \in L_B(n)$ by addition of one grain: $s = s'^{\downarrow i}$ for some integer i. We will now represent this relation by a tree where s is the son of s' iff $s = s'^{\downarrow i}$ and we label with i the edge $s' \longrightarrow s$ in this tree. We denote this tree by $T_B(\infty)$. The root of this tree is (0). The recursive structure of this tree will allow us to obtain a recursive formula for the cardinal of $L_B(n)$.

From the construction of $L_B(n{+}1)$ from $L_B(n)$, it follows that the nodes of this tree are the elements of $\bigsqcup_{n \geq 0} L_B(n)$, and that each node s has at least one son, $s^{\downarrow 1}$, and one more if s begins with a slippery plateau of length l: the element $s^{\downarrow l+1}$. Therefore, $T_B(\infty)$ is a binary tree. We will call *left son* the first of two sons, and *right son* the other (if it exists). We call *the level n of the tree* the set of elements of depth n. The first levels of $T_B(\infty)$ are shown in Figure 5.

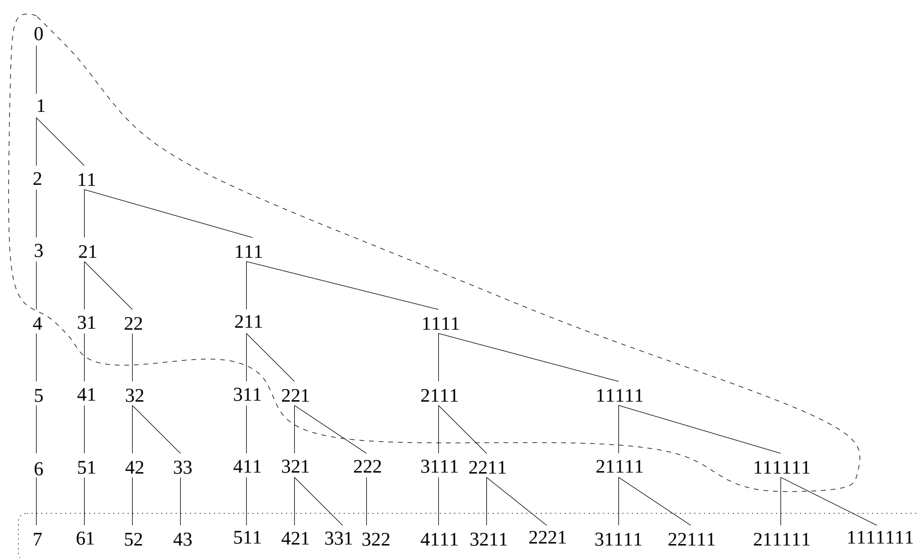

Figure 5: The first levels of the tree $T_B(\infty)$. As shown on this figure for $n = 7$, we will see two ways to find the elements of $L_B(n)$ in $T_B(\infty)$ for any n.

Like in the case of $L_B(\infty)$, there are two ways to find the elements of $L_B(n)$ in $T_B(\infty)$. From the construction of $L_B(n{+}1)$ from $L_B(n)$ given above, it is straightforward that:

Proposition 4 *The level n of $T_B(\infty)$ is exactly the set of the elemens of $L_B(n)$.*

Moreover, it is obvious from the construction of $T_B(\infty)$ that the elements of the set $\overline{L_B(n+1)} \setminus \overline{L_B(n)}$ are sons of elements of $\overline{L_B(n)}$, hence we deduce the following proposition which can be easily proved by induction:

Proposition 5 *Let χ^{-1} be the inverse of the lattice isomorphism defined in Theorem 4. Then, the set $\chi^{-1}(\overline{L_B(n)})$ is a subtree of $T_B(\infty)$ having the same root.*

We will now give a recursive description of $T_B(\infty)$. This will allow us to give a new recursive formula for $|L_B(n)|$, for which no closed formula is known. We first define a certain kind of subtrees of $T_B(\infty)$. Afterwards, we show how the whole structure of $T_B(\infty)$ can be described in terms of such subtrees.

Definition 1 *We will call X_k* subtree *any subtree T of $T_B(\infty)$ which is rooted at an element $s = (\underset{k}{\underleftrightarrow{i, \ldots, i}}, s_{k+1}, \ldots)$ with $s_{k+1} \leq i-1$ and which is either the whole subtree of $T_B(\infty)$ rooted at s if s has only one son, or s and its left subtree if s has two sons. Moreover, we define X_0 as a simple node.*

The next proposition shows that all the X_k subtrees are isomorphic.

Proposition 6 *A X_k subtree, with $k \geq 1$, is composed by a chain of $k+1$ nodes whose edges are labelled 1, 2, $\ldots$, k and whose i-th node is the root of a X_{i-1} subtree for all i between 1 and $k+1$ (See Figure 6).*

Proof : The claim is obvious for $k = 1$. Indeed, in this case the root s has the form $(i, s_2, \ldots)$ with $s_2 \leq i-1$, therefore its left son has the form $(i+1, i-1, \ldots)$, i.e. it starts with a cliff, and has only one son. This son also starts with a cliff; we can then deduce that X_1 is simply a chain, which is the claim where $k = 1$.

Suppose now the claim proved for any $i < k$ and consider the root s of a X_k subtree: $s = (\underset{k}{\underleftrightarrow{i, \ldots, i}}, s_{k+1}, \ldots)$ with $s_{k+1} \leq i-1$. Its left son is $s^{\downarrow_1} = (i+1, i, \ldots, i, s_{k+1}, \ldots)$ with $s_{k+1} \leq i-1$, hence it is the root of a X_1 subtree. Moreover, $s^{\downarrow_1}$ has one right son: $s^{\downarrow_1\downarrow_2} = (i+1, i+1, i, \ldots, i, s_{k+1}, \ldots)$, which by definition is the root of a X_2 subtree. After $k-1$ such stages, we obtain $s^{\downarrow_1\downarrow_2\cdots\downarrow_{k-1}}$, which is equal to $(i+1, \ldots, i+1, i, s_{k+1})$. This node is the root of a X_{k-1} subtree and has a right son: $s^{\downarrow_1\downarrow_2\cdots\downarrow_{k-1}\downarrow_k} = (\underset{k}{\underleftrightarrow{i+1, \ldots, i+1}}, s_{k+1}, \ldots)$, and we still have that $s_{k+1} \leq i-1$. Therefore, this node is the root of a X_k subtree, and from the definition of $T_B(\infty)$ we know that it has no other son. This terminates the proof. $\square$

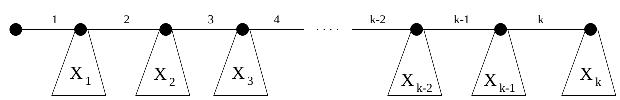

Figure 6: Self-referencing structure of X_k subtrees

This recursive structure and the above propositions allow us to give a compact representation of the tree by a chain:

Theorem 6 *The tree $T_B(\infty)$ can be represented by the infinite chain shown in Figure 7: the i-th node of this chain, $(\underset{i-1}{\underleftrightarrow{1, \ldots, 1}})$, is linked to the following node in the chain by an edge $\xrightarrow{i}$ and is the root of a X_{i-1} subtree.*

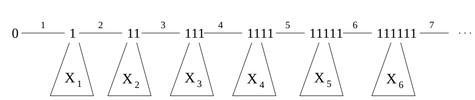

Figure 7: Representation of $T_B(\infty)$ as a chain

Moreover, we can prove a stronger property of each subtree in this chain:

Theorem 7 *The* X_k *subtree of* $T_B(\infty)$ *with root* $(1,\ldots,1)$ *contains exactly the partitions of length* k.

Proof : Because of their recursive structure as shown in Proposition 6, X_k subtrees contain no edge with label greater than k, hence if its root is of length k then all the nodes of a X_k subtree have length k. Moreover, no X_l subtrees with $l \neq k$ and root with length l can contain any node of length k. This remark, together with Theorem 6, implies the result. $\square$

We can now state our last result:

Theorem 8 *Let* $c(l,k)$ *denote the number of paths in a* X_k *tree originating from the root and having length* l. *We have:*

$$c(l,k) = \begin{cases} 1 & \textit{if } l = 0 \textit{ or } k = 1 \\ \sum_{i=1}^{inf(l,k)} c(l-i,i) & \textit{otherwise} \end{cases}$$

Moreover, $|L_B(n)| = c(n,n)$ *and the number of partitions of* n *with length exactly* k *is* $c(n-k,k)$.

Proof : The formula for $c(l,k)$ is derived directly from the structure of X_k trees (Proposition 6 and Figure 6). To obtain $|L_B(n)|$, just remark that it immediately comes from Proposition 6 and Theorem 6 that the two subtrees obtained respectively from $T_B(\infty)$ and X_n by keeping only the nodes of depth at most n are isomorphic. The last formula directly derives from Theorem 6 and 7. $\square$

5 Perspectives

The duplication process that appears during the construction of the lattices of integer partitions may be much more general, and could be extended to other kinds of lattices, maybe leading to the definition of a special class of lattices, which contains the lattices of integer partitions. Moreover, the ideas developped in this paper are very general and could be applied to others dynamical systems, such as Chip Firing Games, or tilings with flips, with some benefit.

References

[And76] George E. Andrews. *The Theory of Partitions*, volume 2 of *Encyclopedia of Mathematics and its applications*. Addison-Wesley Publishing Company, 1976.

[Bry73] T. Brylawski. The lattice of integer partitions. *Discrete Mathematics*, 6:210–219, 1973.

[DP90] B.A. Davey and H.A. Priestley. *Introduction to Lattices and Orders*. Cambridge university press, 1990.

[LMMP98] M. Latapy, R. Mantaci, M. Morvan, and H.D. Phan. Structure of SPM. 1998. Submitted.

A Incremental construction algorithm

Algorithm 1 Incremental construction

Input: $L_B(n)$

Output: $L_B(n+1)$

begin

  $Resu \leftarrow L_B(n)^{\downarrow_1}$;

  add the elements and edges of $S^{\downarrow_2}$, $T^{\downarrow_2}$ and $U_l^{\downarrow_{l+1}}$;

  for each *s in S* **do**

    remove the edge $s^{\downarrow_1} \xrightarrow{1} t^{\downarrow_1}$;

    add the edge $s^{\downarrow_1} \xrightarrow{1} s^{\downarrow_2}$;

    add the edge $s^{\downarrow_2} \xrightarrow{2} t^{\downarrow_1}$;

  for each *s in T* **do**

    add the edge $s^{\downarrow_1} \xrightarrow{1} s^{\downarrow_2}$;

  for each *s in* U_l **do**

    add the edge $s^{\downarrow_1} \xrightarrow{1} s^{\downarrow_{l+1}}$;

    if *s has a cliff at* $l+1$ **then**

      let t such that $s \xrightarrow{l+1} t$ in $L_B(n)$;

      add the edge $s^{\downarrow_{l+1}} \xrightarrow{l+1} t^{\downarrow_{l+1}}$;

    if *s has a slippery step at* l **then**

      let t such that $s \xrightarrow{l} t$ in $L_B(n)$;

      add the edge $s^{\downarrow_{l+1}} \xrightarrow{l+1} t^{\downarrow_l}$;

  Return(Resu);

end

Theorem 9 *Algorithm 1 computes $L_B(n+1)$ from $L_B(n)$ in linear time with respect to the number of added elements and transitions.*

Proof : The fact that Algorithm 1 computes $L_B(n+1)$ from $L_B(n)$ is obvious from Proposition 2 and Theorem 1. To obtain the announced complexity, we must show that the parts S, T and U_l can be found in linear time with respect to their size. Let V denote $S \cup T \cup \bigcup_l U_l$.

Since $L_B(n)$ contains *all* the partitions of n, it contains for every integer p the partition (of n) $s = (p+1, p, \ldots, p, r)$ with $r < p$. All the partitions in S or T with first component $p+1$ are reachable in V from s, since s is the greatest such element with respect to the dominance ordering, and since the only elements directly reachable from an element of $S \cup T$ which are not in $S \cup T$ are those obtained by a transition $\xrightarrow{1}$. Likewise, all the partitions in U_l with first component p are reachable from $s' = (p, \ldots, p, p-1, r)$ with $r < p$. Hence we can compute the part V of $L_B(n)$ by starting from these elements and then compute the part of V they belong to (by a deepfirst search, for example).

Now, the cost of the search of these elements deeply depends on the coding of the graph. However, these elements may be accessible in constant time if one keeps a pointer to them when $L_B(n)$ is computed. For example, if one iterates Algorithm 1 from $L_B(0)$ to obtain $L_B(n)$, it is possible to maintain a pointers list to these elements when they are added, during a previous iteration of the algorithm. With this, one can compute V in linear time with respect to its size, and so the whole computation is linear with respect to the number of added elements and edges. $\square$